GENERALISED ASYMPOTOTICALLY EXACT CONDITION FOR STABILITY OF UNCERTAIN SYSTEMS

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Abstract

Robust stability of polytopic systems is analysed via homogeneous matrix polynomials (HMPs). At each step sufficiency is provided by the solution of a set of relaxed LMIs and test for positive definiteness of certain matrices. Necessity is based on a relaxed generalisation of Polya’s theorem for HMPs.

Key words: homogeneous polynomials, LMI, robust stability, uncertain systems

Introduction. Robustness of linear systems subjected to structured real parametric uncertainty belonging to a compact vector set has been recognised as a key issue in the analysis of control systems. Usually, a quadratic in the state Lyapunov function is postulated, which is either fixed (quadratic stability), or parameter dependent (robust stability). Quadratic stability means stability for any (possibly infinite) time-variation and it provides a simple sufficient condition in terms of LMIs described only at the vertices of a polytope, while robust stability means stability for all possible, but constant values of the uncertainty. The
first one obviously implies the second one, but the converse is not true in general. As a consequence, quadratic stability approaches lead to conservative results, especially if the uncertainty is known to be constant. On the other hand, robust stability cannot be assessed using convex optimisation. In order to reduce the gap between quadratic and robust stability, attempts for reducing the conservatism of LMI methods have been made for more than a decade. Aimed at going beyond parameter-independent Lyapunov functions, LMI techniques were proposed to derive quadratic in the state candidates for Lyapunov functions, which are affine \cite{5,6,11}, quadratic \cite{1} and recently polynomial \cite{2–4,10,14} in the uncertain parameter. Robust stability is verified through convex optimisation problems formulated in terms of parameterised LMIs, which can be efficiently solved by polynomial time algorithms. Necessary and sufficient conditions for the existence of an affine Lyapunov function assuring stability of a polytope were investigated in \cite{10}. A set of LMIs provides sufficient condition for robust stability, while necessity is asymptotically attained as a result of a generalisation of Polya’s theorem to the case of matrix valued functions.

This research work contains three contributions: (i) a new relaxed asymptotically exact condition for positive definiteness of a HMP defined on the unit simplex is suggested, (ii) a new tighter upper bound for the polynomial degree, via which robust stability can be analysed without conservatism, is derived and (iii) the necessary and sufficient LMI condition for robust stability is relaxed and more flexible than these obtained in \cite{6,11}, which may be viewed as particular cases of it.

2. Notations and preliminaries. \( \mathbb{N} \) is the set of positive integers and \( \mathbb{N}_x \) denotes a set of \( x \) positive integers. A real vector \( x \) with \( t \) entries \( x_i \) and a real \( t \times t \) matrix \( X \) with entries \( x_{ij} \) are denoted \( x = (x_i) \in \mathbb{R}^t \) and \( X = [x_{ij}] \in \mathbb{R}_t \). The \( i \)-th eigenvalue of a matrix \( X \) is \( \lambda_i(X) \). The sum of \( N \) non-negative scalars \( s_i \) is \( |s| \). Define also the sets \( x_n \equiv \{ x \in \mathbb{R}^n : x^T x = 1 \} \) and \( \omega_N \equiv \{ \alpha = (\alpha_i) \in \mathbb{R}_N : |\alpha| = 1 \} \).

Consider a HMP in \( \alpha \in \omega_N \) of degree \( k \) with \( \chi(k) = \frac{(k + N - 1)!}{k!(N-1)!}0! = 1 \), symmetric matrix coefficients

\( \Pi(\alpha,k) = \sum_{|k|=k} \alpha_1^{k_1} \alpha_2^{k_2} \ldots \alpha_N^{k_N} P_{k_1 k_2 \ldots k_N} \in \mathbb{R}_n, k_i \in \mathbb{N}_0 \equiv \mathbb{N} \cup 0. \)  

For \( d \in \mathbb{N}_0, d + k = 2\tau \), (1) can be represented as a HMP of an even degree, i. e.,

\( \Pi(\alpha,2\tau) = |\alpha|^d \Pi(\alpha,k) = \sum_{i,j \in \mathbb{N}_N, i \leq j} \bar{\alpha}_i \bar{\alpha}_j P_{ij} = \Pi(\alpha,k) \)

with \( \chi(2\tau) \) matrix coefficients, where \( \bar{\alpha}_i = \alpha_1^{\tau_1} \alpha_2^{\tau_2} \ldots \alpha_N^{\tau_N}, |\tau| = \tau, i = 1,2, \ldots, \) \( \chi(\tau) \), is the \( i \)-th monomial of degree \( \tau \). The monomial \( \bar{\alpha}_i \bar{\alpha}_j \) is said to be even if \( i = j \), otherwise it is called odd.

S. Savov, I. Popchev
Consider the \( \mu \) monomials of degree 2\(|r|\), given by

\[
\tilde{\alpha}_f = \alpha_1^{2r_1} \alpha_2^{2r_2} \ldots \alpha_i^{2r_i} \ldots \alpha_j^{2r_j} \ldots \alpha_N^{2r_N}; \ |r| = 0, 1, \ldots, \tau - 1,
\]

(3)

\[
\mu = \sum_{|r|=0}^{\tau-1} \chi(|r|).
\]

Let \( \alpha(s) = (\alpha_i) \in \mathbb{R}^s, \ 2 \leq s \leq N \), denotes a vector with \( s \) selected entries from \( \alpha \). If \( \nu(s) \) is the set of \( s \times 1 \) real ordered vectors with entries representing a non-descending sequence of scalars, all possible distinct and compatible systems of pairwise inequalities between the entries of \( \alpha(s) \) are described by the finite set of ordered vectors \( \alpha_p(s) \in \nu(s), \ p = 1, 2, \ldots, s! \). Then, any inequality \( \alpha_i \leq \alpha_j, \ i \neq j, \alpha_i, \alpha_j \in \alpha(s), \) implies \( \mu \) monomial inequalities of the form

\[
\tilde{\alpha}_f \alpha_i^{\gamma+1} = \tilde{\alpha}_f \alpha_i^{2r_i} \leq \tilde{\alpha}_f \alpha_i^{\gamma} = \tilde{\alpha}_h \alpha_g; \ f, g, h \in N_{\chi(\tau)}, \ g \neq h,
\]

(4)

where \( \gamma = 2\tau - 2|r| - 1 = 1, 3, 5, \ldots, 2\tau - 1 \) and \( \tilde{\alpha}_f, \tilde{\alpha}_h, \tilde{\alpha}_g \) are some monomials of degree \( \tau \).

Any vector \( \alpha_p(s) \in \nu(s) \) defines \( \mu_s = 0.5s \ (s-1) \) pairwise inequalities (4). Finally, all \( s! \) vectors \( \alpha_p(s) \in \nu(s) \) determine \( s! \mu_s \) systems of such monomial inequalities, which correspond to all possible sets of pairwise inequalities involving the entries of vector \( \alpha(s) \). For a given \( \alpha_p(s) \in \nu(s) \) any odd monomial in (4) serves as an upper bound for at least one even monomial, i. e.,

\[
\tilde{\alpha}_f \alpha_i^{\gamma} \leq \tilde{\alpha}_h \alpha_g; \ f \in N_{\eta_{gh}} \subset N_{\chi(\tau)}, \ \eta_{gh} \geq 1, \ g \neq h
\]

(5)

and any even monomial in (4) is a lower bound for at least one odd monomial.

3. Positive definite HMPs. Consider the following important result concerning HMPs in \( \alpha \in \omega_N \).

**Theorem 3.1.** Let a given HMP in (1) be positive definite. Then, there exists some \( d^* \in \mathbb{N}_0 \), such that for any \( \mu \geq d^* \) all coefficients of the HMP in (2) are positive definite matrices.

Details of the proof may be found, e.g., in [12]. Theorem 3.1 generalises the famous POLYA’s theorem [7] for HMPs and is based on a derived in [9] lower bound for \( d \), which is tighter than the previously obtained ones. It also states an asymptotically exact condition, which provides a systematic way to decide whether a given HMP is positive definite. Unfortunately, this result is a very conservative one with respect to sufficiency. It will be shown that taking into account the monomial inequalities in (4) helps to derive a relaxed and more flexible necessary and sufficient condition for positive definiteness of a HMP in (1).
For a given $s$, consider the associated with some vector $\alpha_p(s) \in \nu(s)$ HMP of degree $2\tau$

$$\tilde{\Pi}_p(\alpha, 2\tau) = \sum_{f,g,h \in N_{\chi(\tau)}, g \neq h} (\bar{\alpha}_f^2 - \bar{\alpha}_h \bar{\alpha}_g) X_{fgh,p} \in \mathbb{R}_n; \ \bar{\alpha}_f^2 \leq \bar{\alpha}_g \bar{\alpha}_h,$$

$$p = 1, 2, \ldots, s!,$$

with $\mu_s$ arbitrary positive semi-definite matrix coefficients $X_{fgh,p}$. Assume that $N$ of the even monomials in (2) are lexically ordered as follows: $\bar{\alpha}_i^2 = \alpha_i^2, i = 1, 2, \ldots, N$. For $\tau > 1$, any of the rest $\chi(\tau) - N$ even monomials can be represented as a product of two distinct monomials of degree $\tau$, i.e., $\bar{\alpha}_i^2 = \bar{\alpha}_u \bar{\alpha}_v, u \in N_{\chi(\tau)}$, which makes possible the definition of the HMP

$$\tilde{\Pi}_0p(\alpha, 2\tau) = \sum_{t = N+1}^{\chi(\tau)} \sum_{u,v \in N_{\chi(\tau)}, u \neq v} (\bar{\alpha}_t^2 - \bar{\alpha}_u \bar{\alpha}_v) X_{tuv,0p} = 0 \in \mathbb{R}_n$$

$$\forall \alpha, \ p = 1, 2, \ldots, s!,$$

with $\chi(\tau) - N$ arbitrary symmetric matrix coefficients. Consider the HMP

$$\Pi_p(\alpha, 2\tau) = \Pi(\alpha, 2\tau) + \tilde{\Pi}_p(\alpha, 2\tau) + \tilde{\Pi}_0p(\alpha, 2\tau) = \sum_{i,j \in N_{\chi(\tau)}, i \leq j} \bar{\alpha}_i \bar{\alpha}_j \Pi_{ij},$$

$$p = 1, 2, \ldots, s!,$$

the associated with it homogeneous scalar polynomial (HSP) in $\alpha$ and $x \in x_n$

$$h_p(\alpha, 2\tau, x) = x^T \Pi_p(\alpha, 2\tau)x = \sum_{i,j \in N_{\chi(\tau)}, i \leq j} \bar{\alpha}_i \bar{\alpha}_j c_{ij,p}(x); \ c_{ij,p}(x) = x^T \Pi_{ij,p}x,$$

$$p = 1, 2, \ldots, s!,$$

and the HSP in $\alpha$

$$h_p(\alpha, 2\tau) = \sum_{i,j \in N_{\chi(\tau)}, i \leq j} \bar{\alpha}_i \bar{\alpha}_j c_{ij,p}; \ c_{ij,p} \leq \Pi_{ij,p}, \ p = 1, 2, \ldots, s!,$$

where $\Pi_{ij,p}$ denotes the respective symmetric matrix coefficient of the HMP in (8), corresponding to the monomial $\bar{\alpha}_i \bar{\alpha}_j$ and vector $\alpha_p(s) \in \nu(s)$. Let $\bar{\alpha}_v \in \mathbb{R}^{\chi(\tau)}$ be the vector containing all monomials $\bar{\alpha}_i$ of degree $\tau$. Consider the quadratic in $\bar{\alpha}_v$ matrix representation of the $p$-th HSP in (10)

$$h_p(\alpha, 2\tau) = \bar{\alpha}_v^T C_p \bar{\alpha}_v; \ C_p = C_p^T \in \mathbb{R}_{\chi(\tau)}, \ p = 1, 2, \ldots, s!.$$
The entries of $C_p$ are the $\chi(2\tau)$ scalar valued coefficients of the HSP and after that $C_p$ is said to be a coefficient matrix (CM) for it.

**Theorem 3.2.** Consider a given HMP in (1). The following statements are equivalent:

(a) there exist scalars $d$ and $c$, such that $d + k = 2\tau$, $1 \leq s \leq N$, $s! [\mu_s + \chi(\tau) - N]$ parameter matrices in (6) and (7) and $s! \chi(2\tau)$ scalars $c_{ij,p}$ defined in (9), such that all $s!$ CMs in (11) are positive definite,

(b) $\Pi(\alpha, k)$ is a positive definite HMP.

**Proof.** Let assertion (a) hold. Consider an arbitrary given vector $\alpha \in \omega_N$. If $s > 1$, then there exists some $p$, such that the $s$ common entries of the vectors $\alpha$ and $\alpha_p(s) \in \nu(s)$ represent the same sequence of non-descending scalars. Having in mind the HMPs in (2), (6), (7), (8), and the HSP in (9), (10), the following matrix and scalar inequalities are valid:

$$\Pi(\alpha, k) = \Pi(\alpha, 2\tau) \geq \Pi_p(\alpha, 2\tau) \iff h_p(\alpha, 2\tau, x) \geq h_p(\alpha, 2\tau) = \tilde{\alpha}^T \bar{C}_p \tilde{\alpha},$$

$$\forall x \in \mathbb{x}_n.$$

Since all CMs are positive definite by assumption and vector $\alpha$ has been arbitrarily chosen, it follows that (1) is a positive definite HMP. If $s = 1$, then $\nu(1)$ is an empty set, $\mu_s = 0$ and $\Pi(\alpha, k) = \Pi_1(\alpha, 2\tau) = \Pi(\alpha, 2\tau) + \Pi_{01}(\alpha, 2\tau)$. Positive definiteness of the single CMC is a sufficient condition for positive definiteness of the HMP. This proves $(a) \Rightarrow (b)$.

Let (b) be valid. According to Theorem 3.1, there exists some $d$, such that all $\chi(2\tau)$ matrix coefficients $P_{ij}$ in (2) are positive definite. Let $X_{tu,0} = 0$, $\forall p = 1, 2, \ldots, s!$, in this case. It will be shown that (a) holds for arbitrary integer $s$. Let $s = 1$. Then, there always exist some appropriate scalars $c_{ij,1}$ in (10), such that $C_1$ is a positive definite CM. Let $s > 1$. Consider an arbitrary vector $\alpha_p(s) \in \nu(s)$ and the associated with it monomial inequality in (5), which implies

$$\sum_{j \in \mathbb{N}_{Ngh}} \tilde{\alpha}^2_j X_{fgh,p} \leq \tilde{\alpha}^T \bar{P}_{gh,p}; \bar{P}_{gh,p} = \sum_{j \in \mathbb{N}_{Ngh}} X_{fgh,p}, X_{fgh,p} \geq 0, g \neq h, p = 1, 2, \ldots, s!$$

Let the coefficient matrices in (6) be chosen, such that $\bar{P}_{gh,p} = P_{gh}$ for all subscript pairs $(g, h)$ in (5). This choice guarantees that $\theta_o(s)$ (number of distinct odd monomials in (4)) matrix coefficient in (8) become $\Pi_{gh,p} = \bar{P}_{gh,p} - P_{gh} = 0$, and $\theta_e(s)$ (number of distinct even monomials in (4)) matrix coefficients are $\Pi_{ff,p} \geq P_{ff}$. The rest of the $\chi(2\tau) - [\theta_o(s) + \theta_e(s)]$ matrix coefficients are not affected and, hence, they remain positive definite. For a given $s$, the integers $\theta_o(s)$ and $\theta_e(s)$ are fixed for all $p = 1, 2, \ldots, s!$. There always exist some appropriate scalars $c_{ij,p}$, such that the respective $p$-th CM is a positive definite one. The vector $\alpha_p(s)$ has been arbitrarily chosen, which proves that $(b) \Rightarrow (a)$ for any $s$.

Theorem 3.2 puts the problem for positive definiteness analysis of HMPs in an LMI framework and it can be viewed as a generalisation of Theorem 3.1. Its aspects will be discussed later on.

4. Robust stability of matrix polytopes. Consider the uncertain linear system \( \dot{x} = A(\alpha)x \), \( A(\alpha) = \sum_{i=1}^{N} \alpha_i A_i \in \mathbb{R}_n \), \( \alpha \in \omega_N \), where all matrices \( A_i \) are fixed and Hurwitz (negative stable). The stability analysis problem for this class of uncertain systems is to determine necessary and sufficient conditions, under which the polytope \( A = \{ A(\alpha) : \alpha \in \omega_N \} \) contains only Hurwitz matrices.

**Theorem 4.1.** [12] The following statements are equivalents:

(a) \( A \) is a Hurwitz polytope;

(b) There exists a valid Lyapunov function \( x^T \Pi(\alpha, k)x \) of degree \( k \leq b + 1 \) for the system.

The significance of this result consists in the determination of a tighter than some previously derived upper bounds for the degree \( k \) of the HMP, via which the robust stability of a given polytopic system can be analysed. E.g., this bound was determined as \( b + n \) in [4] for HMPs and \( 2nN \) in [8], or \( b + n - 1 \) in [14] for general matrix polynomials. Theorem 4.1, assertion (c), provides a necessary and sufficient condition for stability of polytopic systems, expressed in terms of the positive definiteness of a HMP. Theorem 3.2 proposes a new relaxed asymptotically exact condition via which it can be tested. The next theorem summarises them.

**Theorem 4.2.** The following statements are equivalents:

(a) \( \chi(k) \) coefficient matrices in (1), assertion (a), Theorem 3.2, holds for the HMP

\[
\Pi(\alpha, k + d + 1) = -|\alpha|^d [A^T(\alpha)\Pi(\alpha, k) + \Pi(\alpha, k)A(\alpha)]
\]

of degree \( k + d + 1 = 2\tau \);

(b) \( A \) is a Hurwitz polytope.

**Proof.** It is entirely based on Theorem 3.2 and due to self-evidence is omitted.

Theorem 4.1 represents a generalisation of two recent results presented below.

**Theorem 4.3.** [6] Let \( d = 0 \) and \( s = 1 \). An affine HMP in (1) assures robust stability of \( A \) if the single CMC1 in (11) is positive definite.

**Theorem 4.4.** [11] A HMP in (1) assures robust stability of \( A \) if and only if there exists some \( d \), such that all matrix coefficients of the HMP \( |\alpha|^d \Pi(\alpha, k+1) \) (12) are positive definite.

Although Theorem 4.2 and Theorem 4.4 provide distinct, but asymptotically equivalent conditions, their sufficiency parts differ substantially. For the same HMP in (1), via which the robust stability of \( A \) is analysed, the stated conditions are based on Theorems 3.1 and 3.2, respectively. Theorem 4.4 treats the problem as the solution of a rather conservative set of, more all less isolated LMIs, where only the sign of the coefficient matrices is significant. On the contrary, Theorem 4
takes into account their lower bounds and the various relations between them put in a compact matrix form (CM). The proof of Theorem 3.2 shows clearly that if robust stability of $A$ is concluded via Theorem 4.4, the same refers to Theorem 4.2, but not vice versa, since all $s'$ CMs may be positive definite, when some or even all matrix coefficients $\Pi_{ij}$, $i < j$, are not positive definite. Theorem 3.1 is based on the assumption that the uncertainty vector is non-negative, while Theorem 3.2 is aimed at taking some additional advantage from this fact. Taking into account the presence of monomial inequalities makes possible to get some relaxations. E.g., if $P_{ij}$, $i < j$, is a positive semi-definite matrix then this fact can be reflected by the choice $X_{ij,p} = P_{ij}$ for the respective parameter matrix in (7), which leads to $\Pi_{ij,p} = 0$ and $\Pi_{ii,p} = P_{ii} + P_{ij} \geq P_{ii}$, i.e., this approach reflects the contribution of each positive semi-definite term $\bar{\alpha}_i \bar{\alpha}_j P_{ij}$ to the overall HMPs positive definiteness. For $X_{ij,p} = P_{ij} - \lambda_{\min}(P_{ij})I \geq 0$, an advantage is obtained, even if $P_{ij}$ is a sign-indefinite matrix.

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