AGGREGATION OF FUZZY PREFERENCE RELATIONS
BY COMPOSITION

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Abstract
The composition of fuzzy relations for their uniting in multicriteria decision making problems is considered. Some criteria evaluate a set of alternatives by fuzzy preference relations. The comparison between the criteria importance is given as a fuzzy preference relation too. Aggregation of these relations is made with the help of a composition of the couple of the relations by the criteria and then \( t \)-norms and \( t \)-conorms for taking into account the relation between the criteria importance are used. Preservation of the transitivity property of the composition of two relations required to decide the ranking or choice problem is investigated.

Key words: fuzzy preference relations, composition of fuzzy relations, additive transitivity, \( t \)-norms, \( t \)-conorms, aggregation operators, decision making

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1. Introduction. The fuzzy preference relations are the basic concept in the following multicriteria decision making problem. Let \( A = \{a_1, \ldots, a_n\} \) be the finite set of alternatives evaluated by several fuzzy criteria \( K = \{k_1, \ldots, k_m\} \), i.e. these criteria give fuzzy preference relations \( R_1, R_2, \ldots, R_m \) between the alternatives. When the cardinality \( n \) of \( A \) is small, the preference relations may be represented by the \( n \times n \) matrices \( R_k = \|r_{ij}^k\| \), where \( r_{ij}^k = \mu_k(a_i, a_j) \), \( i, j = 1, \ldots, n, \ k = 1, \ldots, m \), \( \mu_k : A \times A \to [0, 1] \) is the membership function of the relation \( R_k \) and \( r_{ij}^k \) is the preference degree of the alternative \( a_i \) over \( a_j \) by the criterion \( k_k \). \( r_{ij}^k = 0.5 \) indicates indifference between \( a_i \) and \( a_j \), \( r_{ij}^k = 1 \) indicates that \( a_i \) is absolutely preferred to \( a_j \).

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and \( r_{ij}^k > 0.5 \) indicates that \( a_i \) is preferred to \( a_j \) by the \( k \)-th criterion. In this case, the preference matrices \( R_k, \ k = 1, \ldots, m \) are usually assumed to be additive reciprocal, i.e.

\[
r_{ij}^k + r_{ji}^k = 1, \ i, j = 1, \ldots, n.
\]

The fuzzy preference relation \( W \) between the criteria is given as well, i.e. the couples of criteria are compared according to their importance. Let \( W = \|w(k_i, k_j)\| = \|w_{ij}\|, \ i, j = 1, \ldots, m \), where \( w(k_i, k_j) \) be the preference degree of the criterion \( k_i \) over \( k_j \). The investigated problem is to obtain the preference relation between the alternatives uniting the fuzzy relations by the individual criteria taking into account the relation between the importance of the criteria. The aim is to use the whole information given above to the final stage of the problem solution.

2. Properties of a composition of fuzzy relations. A method to unite two relations consists in their composition.

Definition [5]. Let \( X \) and \( Y \) be fuzzy relations in \( A \times A \) and let \( \Delta \) be a \( t \)-norm. The composition \( X \circ Y \) of these relations with respect to \( \Delta \) is the fuzzy relation on \( A \times A \) with membership function

\[
(1) \quad \mu(a_i, a_j) = \mu_{X \circ Y}(a_i, a_j) = \max_k \{\mu_X(a_i, a_k) \Delta \mu_Y(a_k, a_j)\}, \ i, j, k = 1, \ldots, n.
\]

When \( t \)-norm \( \Delta = \min \) then the composition is a max-min one. When \( \Delta = xy \), then it is a max-product composition. \( X \circ Y \) can be obtained as the matrix product of \( X \) and \( Y \). It has to be taken into account that \( X \circ Y \neq Y \circ X \).

Let \( X = \|x_{ij}\| \) and \( Y = \|y_{ij}\|, \ i, j = 1, \ldots, n \) be fuzzy relations in \( A \times A \), where \( x_{ij}, y_{ij} \) be the membership degree of the comparison of the alternatives \( a_i, a_j \in A \) to the fuzzy preference relations \( X \) and \( Y \), respectively.

The transitivity is one of the most important properties concerning preferences. The following transitivity properties will be used further on: max-min transitivity, i.e., \( \mu(a, c) \geq \min(\mu(a, b), \mu(b, c)) \), max-\( \Delta \) transitivity, i.e., \( \mu(a, c) \geq \max(0, \mu(a, b) + \mu(b, c) - 1) \), and additive transitivity for a reciprocal relation, i.e., \( \mu(a, c) = \mu(a, b) + \mu(b, c) - 0.5 \), where \( a, b, c \in A, \ \mu : A \times A \to [0,1] \) is the membership function of a fuzzy relation. The interpretation of the additive transitivity and the comparison between it and the other ones is described in \([1,4]\), where a method for constructing fuzzy preference relations from a set of \( (n - 1) \) preference data is given.

Certainly, the properties of the composition of two relations depend on the relations’ properties. The examples show that the composition does not preserve the additive transitivity and the max-min one. But these kinds of transitivity are very strong properties imposed on a fuzzy relation according to \([2,11]\). ZADEH \([11]\) suggested several possible definitions. The weakest of all these definitions is the max-\( \Delta \) transitivity. It is shown that this is the most suitable notion of transitivity for fuzzy ordering. That is why the following proposition proves the conditions under which the composition of two fuzzy relations is a max-\( \Delta \) transitive one.

Proposition 2.1. The composition of two relations is max-\( \Delta \) transitive if the relations are additively transitive.

Proof. Let \( X = \|x_{ij}\| \) and \( Y = \|y_{ij}\|, \ i, j = 1, \ldots, n \) be additively transitive relations, i.e.

\[
(2) \quad x_{ij} = x_{ik} + x_{kj} - 0.5, \ x_{ij} + x_{ji} = 1 \text{ and } y_{ij} = y_{ik} + y_{kj} - 0.5, \ y_{ij} + y_{ji} = 1, \ \forall i, j, k.
\]
Two cases will be considered: max-min composition and max-product composition.

A) MAX-MIN COMPOSITION. In this case if $Z = X \circ Y$, then according to (1)

$$Z = \|z_{ij}\|, \quad z_{ij} = \max_k \{\min(x_{ik}, y_{kj})\}, \quad k = 1, \ldots, n.$$  

It has to be proved that

$$z_{ij} \geq \max(0, z_{ik} + z_{kj} - 1), \quad k = 1, \ldots, n. \quad (4)$$

If $z_{ik} + z_{kj} - 1 \leq 0$, then (4) is proved because $z_{ij} \in [0, 1]$.

Let $z_{ik} + z_{kj} - 1 > 0$, then according to (3) let

$$z_{ik} = \max_s \{\min(x_{is}, y_{sk})\} = \min(x_{ik1}, y_{k1k}) = x_{ik1}, \quad s = 1, \ldots, k1, \ldots, n; \quad (5)$$

$$z_{kj} = \max_s \{\min(x_{ks}, y_{sj})\} = \min(x_{kk2}, y_{k2j}) = x_{kk2}, \quad s = 1, \ldots, k2, \ldots, n; \quad (6)$$

$$z_{ij} = \max_s \{\min(x_{is}, y_{sj})\} = \min(x_{ik3}, y_{k3j}) = x_{ik3}, \quad s = 1, \ldots, k3, \ldots, n. \quad (7)$$

If $x_{ik3} \geq x_{ik1}$ or $x_{ik3} \geq x_{kk2}$, then (4) is proved. Let $x_{ik3} < x_{ik1}$ and $x_{ik3} < x_{kk2}$, but from (7) $\min(x_{ik3}, y_{k3j}) \geq \min(x_{ik1}, y_{k1j})$, therefore, $x_{ik3} \geq \min(x_{ik1}, y_{k1j}) = y_{k1j}$ and it has to be proved that $x_{ik3} \geq y_{k1j} \geq \min(x_{ik1}, y_{k1k}) + \min(x_{kk2}, y_{k2j}) - 1$, but $\min(x_{ik1}, y_{k1k}) + \min(x_{kk2}, y_{k2j}) - 1 \leq y_{k1k} + y_{k2j} - 1$. Hence, if

$$y_{k1j} \geq y_{k1k} + y_{k2j} - 1 \quad (8)$$

is valid, then (4) is valid as well. Taking into account that relation $Y$ is reciprocal, the inequality (8) becomes

$$y_{k1j} \geq y_{k1k} + y_{jk2} \leftrightarrow y_{k1j} + y_{jk2} \geq y_{k1k}. \quad (9)$$

The relation $Y$ is additively transitive and, therefore,

$$y_{k1k2} = y_{k1j} + y_{jk2} - 0.5, \quad \text{i.e.} \quad y_{k1j} + y_{jk2} = y_{k1k} + 0.5,$$

then (9) becomes

$$y_{k1k2} + 0.5 \geq y_{k1k} = 1 - y_{kk3} \leftrightarrow y_{kk1} + y_{k1k2} - 0.5 \geq 0 \leftrightarrow y_{kk2} \geq 0,$$

that is valid.

The proof of the other variants of the minimum values of (5), (6) and (7) is reduced to the above case.

B) MAX-PRODUCT COMPOSITION. In this case if $Z = X \circ Y$, then according to (1)

\[ Z = \|z_{ij}\|, \quad z_{ij} = \max_k \{x_{ik}, y_{kj}\}, \quad k = 1, \ldots, n. \]

Using the above notations it follows:

\[ z_{ik} = \max_s \{x_{is}, y_{sk}\} = x_{ik_1}y_{k_1k}, \quad s = 1, \ldots, k_1, \ldots, n; \]

\[ z_{kj} = \max_s \{x_{ks}, y_{sj}\} = x_{kk_2}y_{k_2j}, \quad s = 1, \ldots, k_2, \ldots, n; \]

\[ z_{ij} = \max_s \{x_{is}, y_{sj}\} = x_{ik_3}y_{k_3j}, \quad s = 1, \ldots, k_3, \ldots, n. \]

Let $z_{ik} + z_{kj} - 1 > 0$ and $z_{ij} < z_{ik}$, $z_{ij} < z_{kj}$, then according to (4), (11), (12) and (13) it has to be proved that

\[ x_{ik_3}y_{k_3j} \geq x_{ik_1}y_{k_1k} + x_{kk_2}y_{k_2j} - 1. \]

It follows from (13) that $z_{ij} = \max_s \{x_{is}, y_{sj}\} = x_{ik_3}y_{k_3j} \geq x_{ik_1}y_{k_1j}$. Then if

\[ x_{ik_1}y_{k_1j} \geq x_{ik_1}y_{k_1k} + x_{kk_2}y_{k_2j} - 1 \]

is valid, (14) will be valid as well.

From $z_{ij} < z_{ik}$ it follows that $x_{ik_1}y_{k_1j} < x_{ik_1}y_{k_1k}$, i.e. $y_{k_1j} < y_{k_1k}$ and then (15) becomes

\[ x_{ik_1}(y_{k_1j} - y_{k_1k}) \geq x_{kk_2}y_{k_2j} - 1 \iff x_{ik_1}(y_{kk_1} + y_{k_1j} - 0.5 - 0.5) \geq x_{kk_2}y_{k_2j} - 1 \iff \]

\[ x_{ik_1}(y_{k_1j} - 0.5) \geq x_{kk_2}y_{k_2j} - 1 \iff 1 - x_{kk_2}y_{k_2j} \geq x_{ik_1}(0.5 - y_{kj}). \]

- If $x_{ik_1} \leq x_{kk_2}$ and from $y_{kj} \leq 0.5 - x_{ik_1}(0.5 - y_{kj}) \leq x_{kk_2}(0.5 - y_{kj})$. Then, it has to be proved that $1 - x_{kk_2}y_{k_2j} \geq x_{kk_2}(0.5 - y_{kj})$, but

\[ 1 - x_{kk_2}y_{k_2j} - x_{kk_2}(0.5 - y_{kj}) = 1 - x_{kk_2}(y_{k_2j} + 0.5 - y_{kj}) = \]

\[ 1 - x_{kk_2}(y_{k_2j} + 0.5 - 1 + y_{jk}) = 1 - x_{kk_2}(y_{k_2j} + y_{jk} - 0.5) = 1 - x_{kk_2}y_{k_2k} \geq 0. \]
• If \( x_{ik_1} > x_{kk_2} \), then \( 1 - x_{kk_2}y_{k_2j} > 1 - x_{ik_1}y_{k_2j} \) and from \( y_{kj} \leq 0.5 \) it has to be proved that

\[
1 - x_{ik_1}y_{k_2j} \geq x_{ik_3}(0.5 - y_{kj}) \Leftrightarrow 1 - x_{ik_1}y_{k_2j} - x_{ik_1}(0.5 - y_{kj}) =
\]

\[
1 - x_{ik_1}(y_{k_2j} + 0.5 - y_{kj}) = 1 - x_{ik_1}y_{k_2j} \geq 0.
\]

The proof for the transitivity of the composition \( Y \circ X \) is done similarly.

3. Model for aggregation of fuzzy relations with different importance.

The following model for aggregation of the fuzzy relations by the above initial information is suggested here. Let \( X = \|x_{ij}\| \) and \( Y = \|y_{ij}\| \), \( i, j = 1, \ldots , n \) be fuzzy relations in \( A \times A \), where \( x_{ij}, y_{ij} \) are the membership degree of the comparison of the alternatives \( a_i, a_j \in A \) to the fuzzy preference relations \( X \) and \( Y \), respectively. Taking into account the relation \( W \) between the criteria giving these relations, a new fuzzy relation \( R \) between \( X \) and \( Y \), \( X \neq Y \) with the following membership degrees is suggested:

\[
(16) \quad r_{ij} = \begin{cases} 
0.5 & \text{if } a_i = a_j, \\
S(T(w^1, z_{ij}^1), T(w^2, z_{ij}^2)) & \text{if } a_i \neq a_j,
\end{cases}
\]

where \( Z^1 = X \circ Y, Z^2 = Y \circ X \) are compositions from (1), \( w^1 = w(k_X, k_Y) \), \( w^2 = w(k_Y, k_X) \) are the preference degrees of the criterion with a relation \( X \) over \( Y \) and \( Y \) over \( X \), respectively; and \( T \) is a \( t \)-norm and \( S \) is a corresponding \( t \)-conorm.

The main idea used in (16) is that the composition of two relations compares the preference degrees of the \( i \)-th alternative to all alternatives from the first relation with the preference degrees of the all alternatives to the \( j \)-th alternative from the second relation and vise versa, because the operation composition is not commutative. Then taking into account the relation \( W \), i.e. that \( w^1 \) and \( w^2 \) are the preference degree of the relation \( X \) over \( Y \) and \( Y \) over \( X \), respectively, a comparison operator is used (\( t \)-norm) that “pessimistically” combines the relations \( W \) and \( Z^1 \), \( W \) and \( Z^2 \) to obtain measures of match which can be after that “optimistically” united by \( t \)-conorm in an overall result \([6]\). If the fuzzy relations corresponding to the criteria are \( R_1, R_2, \ldots , R_m \), the new fuzzy relation according to (16) is \( R_{ij} \) and as \( R_{ij} = R_{ji} \), the number \( k \) of the new relations will be equal to the combinations of two elements over \( m \), i.e. \( k = \frac{m(m-1)}{2} \).

Aggregation operators \([7]\) uniting these \( k \) relations can be used after that to obtain the aggregation fuzzy relation giving a possibility to decide the choice or ranking problems.

The following proposition proves that the relation (16) preserves the property of max-\( \Delta \) transitivity of the compositions \( Z^1 = \|z_{ij}^1\| \) and \( Z^2 = \|z_{ij}^2\| \).

**Proposition 3.1.** The relation (16) is max-\( \Delta \) transitive for the couple of \( t \)-norms \( (T = \min, S = \max) \) and \( (T = xy, S = x + y - xy) \) if the relations \( Z^1, Z^2 \) are max-\( \Delta \) transitive ones and the relation \( W \) is additive reciprocal.

**Proof.** Let \( Z^1, Z^2 \) be max-\( \Delta \) transitive relations, i.e.

\[
(17) \quad z_{ij}^1 \geq \max(0, z_{ik}^1 + z_{kj}^1 - 1), \quad z_{ij}^2 \geq \max(0, z_{ik}^2 + z_{kj}^2 - 1), \quad k = 1, \ldots , n
\]
and $W$ is additive reciprocal, i.e.

$$w^1 + w^2 = 1. \tag{18}$$

Then it has to be proved that

$$S(T(w^1, z_{ij}^1), T(w^2, z_{ij}^2)) \geq \max(0, S(T(w^1, z_{ik}^1), T(w^2, z_{ik}^2)) + S(T(w^1, z_{kj}^2), T(w^2, z_{kj}^2)) - 1). \tag{19}$$

The following notations will be used for simplicity:

$$z_{ij}^1 = r, \quad z_{ij}^2 = q, \quad z_{ik}^1 = a, \quad z_{ik}^2 = b, \quad z_{kj}^1 = c, \quad z_{kj}^2 = d, \quad w^1 = x, \quad w^2 = y.$$

Then (17), (18) and (19) may be rewritten as

$$r \geq \max(0, a + b - 1), \quad q \geq \max(0, c + d - 1), \quad x + y = 1; \tag{20}$$

$$S(T(x, r), T(y, q)) \geq \max(0, S(T(x, a), T(y, b)) + S(T(x, c), T(y, d)) - 1). \tag{21}$$

If $S(T(x, r), T(y, q)) \geq S(T(x, a), T(y, b))$ or $S(T(x, r), T(y, q)) \geq S(T(x, c), T(y, d))$, or $S(T(x, a), T(y, b)) + S(T(x, c), T(y, d)) - 1 \leq 0$ then (21) is valid. For the other cases it has to be proved that

$$S(T(x, r), T(y, q)) \geq S(T(x, a), T(y, b)) + S(T(x, c), T(y, d)) - 1. \tag{22}$$

The most complicated case will be considered taking into account (20). Let

$$0 \leq r \leq a \leq c \leq 1, \quad 0 \leq q \leq b \leq d \leq 1, \quad 0 \leq x \leq y \leq 1. \tag{23}$$

A) Let $T = \min$, $S = \max$. In this case from (22) it has to be proved that

$$\max(\min(x, a), \min(y, b)) + \max(\min(x, c), \min(y, d)) - 1 \leq \max(\min(x, r), \min(y, q)). \tag{24}$$

The 16 variants are possible taking into account (23), the position of $x$ between $r, a, c$ and the position of $y$ between $q, b, d$, for example:

- Let $x \leq r, y \leq q$, then (24) becomes $y + y - 1 \leq y$, which is valid.
- Let $x \leq r, q \leq y \leq b$, then (24) becomes $\max(x, y) + \max(x, y) - 1 \leq \max(x, q)$,
  - if $\max(x, q) = x$, then it has to be proved that $y + y - 1 \leq x$, but according to (23) and (20) $y + y - 1 \leq b + d - 1 \leq q \leq x$,
  - if $\max(x, q) = q$, then (24) becomes $y + y - 1 \leq q$, that is valid.

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• Let $a \leq c \leq b \leq y \leq d$, then (24) becomes

$$\max(a, b) + \max(x, y) - 1 \leq \max(r, q),$$

– if $a \leq b$, $r \leq q$, then one has according to (20) $b + y - 1 \leq b + d - 1 \leq q,$
– if $a \leq b$, $r > q$, then $b + y - 1 \leq b + d - 1 \leq q \leq r,$
– if $a > b$, $r \leq q$, then $x + y - 1 \leq 1 - 1 = 0 \leq q,$
– if $a > b$, $r > q$, then $x + y - 1 \leq r.$

• Let $c \leq x \leq 1, \quad d \leq y \leq 1$, then (24) becomes $\max(a, b) + \max(c, d) - 1 \leq \max(r, q)$, but $\max(a, b) + \max(c, d) \leq x + y \leq 1$ and, hence, the above inequality is valid.

Therefore, (19) is proved for $T = \min, \quad S = \max.$

B) Let $T = xy, \quad S = x + y - xy$. In this case it has to be proved that (from (22))

$$x(r + y - qy) - 1 \geq x + y - xab + xc + yd - xycd - 1. \tag{25}$$

After simple transformations and the condition (20), (25) becomes

$$x(r - a - c + 1) + y(q - b - d + 1) + xy(ab + cd - rq) \geq 0. \tag{26}$$

Let $0 \leq r \leq a \leq c \leq 1, \quad 0 \leq q \leq b \leq d \leq 1$, then $rq \leq ab \leq cd$ and (25) is valid.

Let $0 \leq a \leq r \leq c \leq 1, \quad 0 \leq q \leq b \leq d \leq 1$, then $rq \leq cd$ and (25) is valid.

Let $0 \leq a \leq r \leq c \leq 1, \quad 0 \leq b \leq q \leq d \leq 1$, then $S(T(x, r), T(y, q)) \geq S(T(x, a), T(y, b))$ and (22) is valid.

The other cases reduce to the last one.

Therefore (19) is proved for these $t$-norms and corresponding $t$-conorms. The other $t$-conorms do not preserve the $\max-\Delta$ transitivity of fuzzy relations as it is proved in $[9]$.  

5. Concluding remarks. The composition of two fuzzy relations is used to aggregation of the relations. A combination of $t$-norm and $t$-conorm is studied for obtaining a new fuzzy preference relation including the computed compositions. This relation connects the individual relations with the relation between the importances of the fuzzy criteria evaluating the set of alternatives. It is proved that the composition of two relations is $\max-\Delta$ transitive relation if the initial relations are additively transitive. The new relation preserves the $\max-\Delta$ transitivity of the composition of the relations under the defined conditions which gives a possibility to decide the problem of ordering the set of alternatives. The main difficulty lies in the number of the computation increase which may be significant, if the number of alternatives and criteria is large. The advantage is that the whole information is used to the final step of problem solving and the new relation has the property of transitivity, that it is important to solve the problems of ordering and ranking of the set of alternatives.

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